MATH-SHU 350

Recitation 7: Martingale

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Exercise 1. *Recall the following definitions in the course: filtration, adapted process, martingale (submartingale, supermartingale), previsible process, stopping time.*

Exercise 2. *Prove that if $(X_n)_{n\geqslant 0}$ is a martingale, then $(|X_n|)_{n\geqslant 0}$ is a submartingale.*

Exercise 3. *Let $(X_n)_{n\geqslant 0}$ be i.i.d. random variables with mean μ and variance $\sigma^2 < \infty$. Let $\mathcal{F}_n = \sigma(X_1, \cdots, X_n)$ be its natural filtration. We define that $S_n := \sum_{i=1}^n X_i - \mu n$. Show that:*

1. *$(S_n)_{n\geqslant 0}$ is a martingale;*
2. *If $\mu = 0$, then $(S_n)^2 - \sigma^2 n$ is a martingale;*
3. *We define $\tau_M := \inf\{n \in \mathbb{N} : S_n \geqslant M\}$. Prove that τ_M is a stopping time.*

Exercise 4. *Let $(X_n)_{n\geqslant 2}$ be a sequence of independent random variables satisfying*

$$\mathbb{P}[X_n = k] = \binom{n}{k}\left(\frac{1}{n}\right)^k\left(1 - \frac{1}{n}\right)^{n-k},$$

for $k \in \{0, \cdots, n\}$. Show that

1. *$\tau = \inf\{n \in \mathbb{N} : X_n > 1\}$ is a stopping time with respect to the filtration $\mathcal{F}_n = \sigma(X_2, \cdots, X_n)$.*
2. *τ is almost surely finite.*

Exercise 5. *Suppose that $(X_n)_{n\geqslant 0}$ is a submartingale and τ is a stopping time with $\mathbb{P}[\tau \leqslant k] = 1$. Show that*

$$\mathbb{E}[X_0] \leqslant \mathbb{E}[X_\tau] \leqslant \mathbb{E}[X_k].$$